

Truth and Logical Consequence

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Consequentia 'formalis' vocatur quae in omnibus terminis valet retenta forma consimili. Vel si vis expresse loqui de vi sermonis, consequentia formalis est cui omnis propositio similis in forma quae formaretur esset bona consequentia [...]

John Buridan, *Tractatus de Consequentibus*, ca. 1335

(Hubien, 1976, I.3, p.22f)

formal validity

All men are mortal. Socrates is a man. Therefore Socrates is mortal.

analytic validity

John is a bachelor. Therefore John is unmarried.

metaphysical validity

There is H₂O in the beaker. Therefore there is water in the beaker.

Logical validity is formal validity (but see, e.g, Read 1994).

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I concentrate on first-order languages, but with strong axioms.

Terminology:

- Sentences and arguments can be logically valid.
- A sentence is logically true iff it's logically valid.
- A conclusion follows logically from (is a logical consequence of) premisses iff the argument is valid.

I concentrate on logical truth; but everything applies *mutatis mutandis* to logical consequence.

model-theoretic definition of validity

A sentence is logically valid iff it's true in all models.

Problems with the model-theoretic definition:

- ▶ Model-theoretic validity doesn't obviously imply truth.
- ▶ Model-theoretic consequence doesn't obviously preserve truth.
- ▶ Model-theoretic validity doesn't obviously imply 'intuitive' validity; it isn't obviously sound.

inferentialist definition of validity

A sentence is logically valid iff it's provable in the system X , e.g., Gentzen's Natural Deduction.

Problems with the inferentialist definition:

- ▶ The inferentialist analysis requires arguments why the rules aren't accidental.
- ▶ 'Intuitive' validity doesn't obviously imply inferentialist validity.
- ▶ Truth preservation isn't built into the definition inferentialist validity.

These observations suggest that neither the inferentialist nor the model-theoretic definition is an adequate analysis of logical validity, even though they may be extensionally correct.

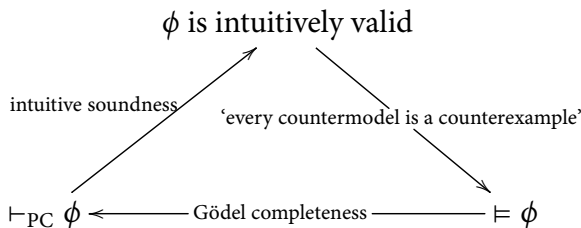
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Kreisel (1965, 1967) argued for the extensional correctness of the model-theoretic definition for first-order logic with his *squeezing argument*:

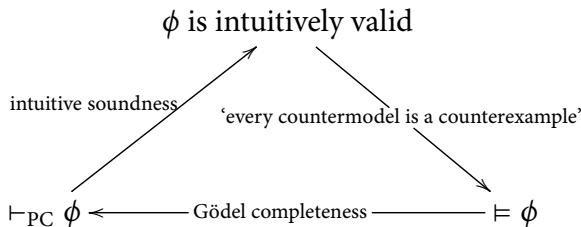
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Back to Buridan and the naive formality conception!

Rough idea for definition:

A sentence is logically valid iff all its substitution instances are true.

A substitution instance is obtained by uniformly replacing predicate symbols with suitable formulae etc.

In what follows I make this idea *formally* precise. The resulting substitutional definition of validity can replace the intuitive informal notion of validity. We can then – arguably – dispense with informal rigour and just prove theorems.

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I speculate that the reasons for the demise of the substitutional account are the following:

- ▶ Tarski's distinction between object and metalanguage in (Tarski, 1936a,b)
- ▶ set-theoretic reductionism (especially after Tarski and Vaught 1956) and resistance against primitive semantic notions
- ▶ usefulness of the set-theoretic analysis for model theory
- ▶ availability of 'squeezing' arguments (even before Kreisel)

Rough idea: *A sentence is logically valid iff all its substitution instances are true.*

Required notions:

- ‘substitution instance’: substitutional interpretations
- ‘true’: axioms for satisfaction

Here are some substitution instances of the *modus barbara* argument:

original argument

All men are mortal. Socrates is a man. Therefore Socrates is mortal.

substitution instance

All wise philosophers with a long beard are mortal. Socrates is a wise philosopher with a long beard. Therefore Socrates is mortal.

substitution instance

All starfish live in the sea. That (animal) is a starfish. Therefore that (animal) lives in the sea.

substitution instance

All objects in the box are smaller than that (object). The pen is in the box. Therefore it is smaller than that (object).

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Setting: I start from a first-order language that is an extension of set theory, possibly with urelements.

Assume the language has only predicate symbols as nonlogical symbols:

A substitutional interpretation is a function that replaces uniformly every predicate symbol in a formula with some formula and possibly relativizes all quantifiers (variables may have to be renamed).

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Assume we have a binary symbols R , S and unary symbols P and Q in the language.

$\forall x (Px \rightarrow \exists y Ryx)$ (original formula)

$\forall x (Qx \rightarrow \exists y \neg \exists z (Qy \wedge Sxz))$ (substitution instance)

$\forall x (Rxx \rightarrow \exists y Ryx)$ (original formula)

$\forall x (\underline{Rzx} \rightarrow (Qx \rightarrow \exists y (\underline{Rzx} \wedge \neg Ryx)))$ (substitution instance)

The underlined formula is the relativizing formula.

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The relativizing formula may be $Px \wedge \neg Px$ (or even $P \wedge \neg P$). This corresponds to the empty domain in model-theoretic semantics and we'll obtain a notion of logical consequence in free logic.

To get standard classical logic I add the following antecedent to each substitution instance with $R(x)$ as relativizing formula:

$$\exists x Rx \wedge Ra_1 \wedge \dots \wedge Ra_n \rightarrow$$

where $a_1, \dots, Ra_n$ are all the individual constants in the substitution instance. This excludes the counterpart of nondenoting constants.

The axioms for Sat are added to our overall theory, an extension of ZF. Syntax is coded as usual.

$$\forall a \forall v \forall w \left(\text{Sat}(\ulcorner Rvw \urcorner, a) \leftrightarrow Ra(v)a(w) \right)$$

and similarly for predicate symbols other than R or Sat

$$\forall a \forall \phi \left(\text{Sat}(\ulcorner \neg \phi \urcorner, a) \leftrightarrow \neg \text{Sat}(\ulcorner \phi \urcorner, a) \right)$$

$$\forall a \forall \phi \forall \psi \left(\text{Sat}(\ulcorner \phi \wedge \psi \urcorner, a) \leftrightarrow (\text{Sat}(\ulcorner \phi \urcorner, a) \wedge \text{Sat}(\ulcorner \psi \urcorner, a)) \right)$$

$$\forall a \forall v \forall \phi \left(\text{Sat}(\ulcorner \forall v \phi \urcorner, a) \leftrightarrow \forall b \left('b \text{ is } v\text{-variant of } a' \rightarrow (\text{Sat}(\ulcorner \phi \urcorner, b)) \right) \right)$$

Schemata of the base theory are extended to the language with Sat .

The quantifiers for a and b range over variables assignments, the quantifiers for ϕ and ψ over formulae of the *entire* language (including Sat).

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‘A sentence is logically valid iff all its substitution instances are true.’

substitutional definition of logical validity

$$\forall \phi \left(\text{Val}(\ulcorner \phi \urcorner) :\leftrightarrow \forall I \forall a \text{Sat}(I\ulcorner \phi \urcorner, a) \right)$$

Here I ranges over substitutional interpretations.

A sentence is defined to be true iff it's satisfied by all variable assignments, i.e., iff $\forall a \text{Sat}(\ulcorner \phi \urcorner, a)$

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Similarly, an argument is logically valid iff there is no substitutional interpretation that makes the premisses true and the conclusion false.

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From this definition we'll get a notion of logical validity in free logic, because the relativizing formula in a substitutional interpretation may not be satisfied by any object.

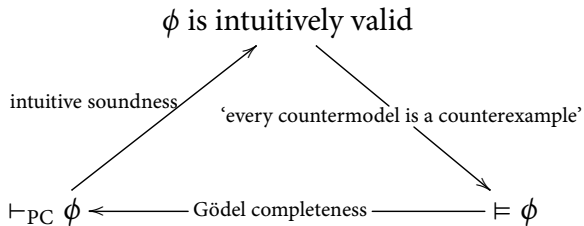
If we are after classical validity with no empty domains, we can explicitly quantify over substitutional interpretations that have a relativizing formula that applies to at least one object under the given variable assignment.

We can also force negative free logic etc by tweaking the definition of substitutional interpretations.

The identity function with $I(\phi) = \phi$ on sentences is a substitutional interpretation. Thus we have:

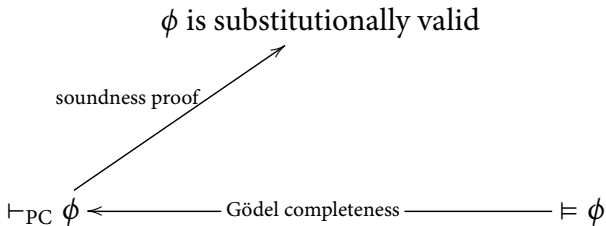
- ▶ Logical validity (logical truth) trivially implies truth, i.e.,
 $\forall x (\text{Val}(x) \rightarrow \forall a \text{Sat}(x, a))$.
- ▶ Similarly, logical consequence preserves truth.

On the substitutional account, the ‘intended’ interpretation is completely trivial. No need for mysterious intended models (whose existence is refutable) or indefinite extensibility.

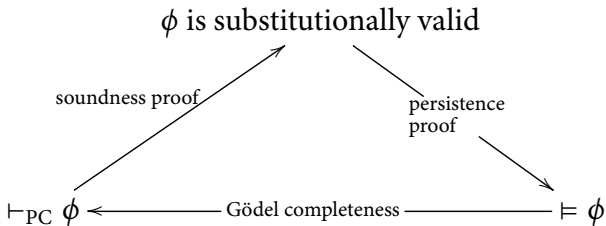


ϕ is substitutionally valid

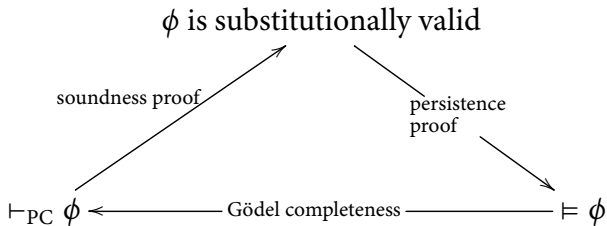
$\vdash_{\text{PC}} \phi \longleftarrow \text{Gödel completeness} \longrightarrow \models \phi$



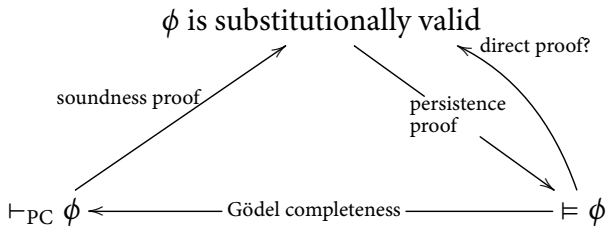
One can prove in the theory that logical provability implies truth under all substitutional interpretations (modulo tweaks concerning the empty domain).



Every set-theoretic countermodel corresponds to a substitutional countermodel.



Now we have ‘squeezed’ substitutional validity.



For this direction I don't know a direct proof. We would need 'Löwenheim–Skolem downwards' for classes.

The squeezing argument for substitutional validity naturally slots into the place of Kreisel's (1967) 'intuitive logical validity'.

Main Thesis: **Logical validity is substitutional validity.**

The traditional 'intuitive' notion requires an absolute satisfaction predicate.

The substitutional notion of validity isn't very sensitive to the choice of nonlogical expressions, if free variables are admitted in the substitution instances.

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- ▶ nonclassical logics: e.g., K3; use PKF
- ▶ constant domain semantics; see Williamson (2000)
- ▶ identity as logical constant
- ▶ second-order logic

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Worries:

- ▶ Why not use a primitive notion of validity instead of the primitive notion Sat (cf. Field 2015)?
- ▶ Why should we tie logical validity to a primitive and problematic notion of satisfaction? Why not use just model-theoretic validity?
- ▶ Isn't it problematic to apply the substitutional account to arbitrary formal languages? Tarski (1936b) was able to apply his account to weak languages.
- ▶ Is logical validity universal on the substitutional account?
- ▶ Doesn't the liar paradox threaten Sat and thereby substitutional validity?

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